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Early cosmological models with variable G and zero-rest-mass scalar fields

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Abstract : Einstein's field equations for zero-curvature Robertson-Walker model of the universe with variable gravitational 'constant' G and zero-rest-mass scalar fields are considered in which the perfect fluid satisfies the 'gamma-law' equation of state $p = (\gamma - 1)\rho$. The γ-index describing the material content varies continuously with cosmological time and this allows a unified description of the early evolution of universe. The solutions of the field equations are presented for the inflationary phase and the radiation-dominated phase. Some physical properties of the cosmological models are also discussed

Keywords : Early universe, scalar fields, cosmological parameters

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1. Introduction

In general relativity, the constant of gravity G plays the role of a coupling constant between geometry and matter in Einstein's field equations. The value of G has to be constant since G-constancy is in-built as a manifestation of the principle of equivalence. A breakdown from the principle of equivalence, in any form, would constitute a departure from Einstein's general relativity. There are several extensions of Einstein's theory of gravitation in which G is taken to vary with cosmic time [1]. The time-dependent G follows as a natural consequence of Dirac's large number hypothesis [2]. The implication of time-varying G will become important only at the early stage of the evolution of the universe. It appears natural to look at this constant as a function of time in an evolving universe. A large body of literature can be found on the evolving universe with matter satisfying the equation of state, $p = (\gamma - 1)\rho$, $1 \leq \gamma \leq 2$.

Israelit and Rosen [3] have obtained a singularity-free model of the evolving universe with matter and studied the transition from the inflationary to radiation-dominated and matter-dominated periods of the universe by using an equation of state. Recently, Carvalho [4] has studied a homogeneous and isotropic cosmological model in which the parameter gamma of 'gamma-law' equation of state $p = (\gamma - 1)\rho$, varies continuously with cosmic time t. He studied the evolution of the universe as it goes from an inflationary phase to a radiation-dominated phase.

In this paper, we study the evolution of universe with the zero-curvature Robertson-Walker models in the presence of zero-rest-mass scalar fields in which the gravitational parameter G varies with cosmic time t. Solutions are obtained for inflationary phase and radiation-dominated phase by using the equation of state, suggested by Carvalho [4]. The physical behaviour of the cosmological solutions are also discussed.

2. Field equations

We consider the homogeneous and isotropic Robertson-Walker line-element

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right], \tag{1}$$

where $R(t)$ is the scale factor and k is the curvature index which takes values +1, 0, −1 for the spaces of positive, vanishing and negative curvature respectively.

The Einstein field equations for matter coupled with a zero-rest-mass scalar field are

$$R_{ij} - \frac{1}{2}g_{ij}R = -8\pi G(t)\left[T_{ij} + S_{ij}\right], \tag{2}$$

where g_{ij} is the metric tensor, R_{ij} the Ricci-tensor, R the scalar curvature, T_{ij} is the energy-momentum tensor of matter field and S_{ij} the energy-momentum tensor for a zero-rest-mass scalar field given by [5]

$$S_{ij} = \frac{1}{4\pi}\left[V_{,i}V_{,j} - \frac{1}{2}g_{ij}V_{,\alpha}V^{,\alpha}\right], \tag{3}$$

where the scalar potential V satisfies

$$g^{ij}V_{;ij} = 0. \tag{4}$$

For a perfect fluid distribution, the energy-momentum tensor T_{ij} is of the form

$$T_{ij} = (\rho + p)u_i u_j - pg_{ij}, u_i u^i = 1, \tag{5}$$

where p is the pressure, ρ, the matter energy-density and u^i the four-velocity vector. A comma and a semi-colon denotes ordinary and covariant differentiation respectively.

In comoving coordinates system, the field equations (2) – (5), for the metric (1), lead to the following equations

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = -8\pi G(t)p - G(t)\dot{V}^2 \tag{6}$$

and

$$3\frac{\dot{R}^2}{R^2} + \frac{3K}{R^2} = 8\pi G(t)\rho + G(t)\dot{V}^2. \tag{7}$$

An overdot denotes differentiation with respect to t. Eqs. (6) and (7) can be rewritten as

$$\frac{\ddot{R}}{R} = -\frac{4}{3}\pi G(t)(\rho+3p) - \frac{2}{3}G(t)\dot{V}^2 \tag{8}$$

and

$$R\ddot{R} + 2(R^2+k) = 4\pi G(t)(\rho-p)R^2. \tag{9}$$

Eliminating $\ddot{R}$ from (8) and (9), we get

$$\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = \frac{8}{3}\pi G(t)\rho + \frac{1}{3}G(t)\dot{V}^2. \tag{10}$$

Eq. (4) gives

$$\ddot{V} + \frac{3\dot{R}}{R}\dot{V} = 0. \tag{11}$$

Eqs. (8) and (10) can be written in terms of Hubble parameter $H = \dot{R}/R$ as

$$\dot{H} + H^2 = -\frac{4}{3}\pi G(t)(\rho+3p) - \frac{2}{3}G(t)\dot{V}^2 \tag{12}$$

and

$$H^2 + \frac{k}{R^2} = \frac{8}{3}\pi G(t)\rho + \frac{1}{3}G(t)\dot{V}^2. \tag{13}$$

In order to solve the above equations, we assume that the pressure p and energy-density ρ are related through the 'gamma-law' equation of state

$$p = (\gamma - 1)\rho, \tag{14}$$

where the adiabatic parameter γ varies continuously with cosmic time during the phase transition from an inflationary phase to a radiation-dominated phase of the universe. Carvalho [4] assumed the parameter γ of the form

$$\gamma(R) = \frac{4}{3}\,\frac{A(R/R_*)^2 + (a/2)(R/R_*)^a}{A(R/R_*)^2 + (R/R_*)^a} \tag{15}$$

where A is constant and parameter a is related to the power of the cosmic time t during an inflationary era and lies in the range $0 \le a < 1$. The function $\gamma(R)$ is such that when the scale factor $R(t)$ is less than a certain reference value R_*, we have the inflationary phase ($\gamma < 2/3$). As the scale factor increases, γ also increases to reach the value 4/3 for $R \gg R_*$ and thus we have the radiation-dominated era.

Substituting the value of p from (14) into (12), we get

$$\dot{H} + H^2 = -\frac{8}{3}\pi G(t)\left(\frac{3}{2}\gamma - 1\right)\rho - \frac{2}{3}G(t)\dot{V}^2. \tag{16}$$

Eliminating ρ between eqs. (13) and (16), we obtain

$$\dot{H} + \frac{3}{2}\gamma H^2 + \left(\frac{3}{2}\gamma - 1\right)\frac{k}{R^2} - \left(\frac{1}{2}\gamma - 1\right)G(t)\dot{V}^2 = 0. \tag{17}$$

To solve eq. (17), we rewrite it in the form

$$HH' + \frac{3}{2}\gamma\frac{H^2}{R} + \left(\frac{3}{2}\gamma - 1\right)\frac{k}{R^3} - \left(\frac{1}{2}\gamma - 1\right)\frac{G(t)\dot{V}^2}{R} = 0, \tag{18}$$

where a dash (′) denotes differentiation with respect to R. For zero curvature Robertson-Walker model ($k = 0$), eq. (18) takes the form

$$H' + \frac{3}{2}\gamma\frac{H}{R} - \left(\frac{1}{2}\gamma - 1\right)G(t)\frac{\dot{V}^2}{HR} = 0. \tag{19}$$

An additional equation relating the time changes of G can be obtained by the Bianchi identities $\left(R_{ij} - \frac{1}{2}R g_{ij}\right)^{;j} = 0 = \left\{8\pi G\left(T_{ij} + S_{ij}\right)\right\}^{;j}$, which yield

$$\left(\rho + \frac{1}{8\pi}\dot{V}^2\right)^{\cdot} + \left\{(\rho + p) + \frac{1}{4\pi}\dot{V}^2\right\}3\frac{\dot{R}}{R} + \left(\rho + \frac{1}{8\pi}\dot{V}^2\right)\frac{\dot{G}}{G} = 0. \tag{20}$$

3. Solution of the field equations

Eq. (11) has the first integral

$$\dot{V} = l/R^3, \tag{21}$$

where l is the integration constant. Using eq (21) into (19), we obtain

$$H' + \frac{3}{2}\gamma\frac{H}{R} - \left(\frac{1}{2}\gamma - 1\right)\frac{l^2 G(t)}{HR^7} = 0. \tag{22}$$

Eq. (22), involving two arbitrary functions $R(t)$ and $G(t)$, admits solution only if one of these is specified. In most of the variable G cosmologies, G is a decreasing function of time [6,7]. The possibility of an increasing G has also been suggested by Levit [8] Beesham [9] has discussed the possibility of the creation field with $G \propto t^n$. Sistero [10] has presented exact solutions for zero pressure Robertson-Walker cosmological models with $G \propto R^n$. For mathematical convenience, we assume the time-dependent G of the form

$$G(t) = m(HR^3)^2, \tag{23}$$

where m being a positive constant. Using eq. (23) into (22), we obtain

$$H' + \left[\left(\frac{3}{2} - \frac{1}{2}\lambda\right)\gamma + \lambda\right]\frac{H}{R} = 0, \tag{24}$$

where $\lambda = l^2 m$ is another positive constant. On integration of eq. (24), we get

$$H = \frac{C}{R^\lambda\left[A(R/R_*)^2 + (R/R_*)^a\right]^{(3-\lambda)/3}}, \tag{25}$$

where C is the integration constant. If $H = H_*$ for $R = R_*$, a relation between C and A can be written in the form

$$C = H_*[1 + A]^{(3-\lambda)/3} R_*^\lambda. \tag{26}$$

By use of equation (26) into (25), an expression for t in terms of scale factor R can be written as

$$H_*(1+A)^{(3-\lambda)/3}t = \int (R/R_*)^{\lambda}\,\frac{\left[A(R/R_*)^2 + (R/R_*)^a\right]^{(3-\lambda)/3}}{R}\,dR \cdot \quad (27)$$

During the course of evolution, the deceleration parameter is not constant and its value for any cosmological time can be calculated from eq. (24) to give

$$q = \left[(3-\lambda)/3\right](3\gamma/2) + \lambda - 1, \quad (28)$$

which clearly depends upon R via γ.

We solve the eq. (27) for inflationary phase and radiation-dominated phase separately in the following sections starting with the inflationary phase.

3.1 Inflationary phase :

When we consider the inflationary phase ($R \ll R_*$), the second term inside the square bracket on right-hand side of integral (27) dominates over the first term which gives a phase of power law inflation for $0 < a < 1$. The scale factor R for $[3a + (3-a)\lambda] \neq 0$, is given by

$$R = R_*\left[\frac{[3a + (3-a)\lambda]}{3} H_*(1+A)^{(3-\lambda)/3}\,t\right]^{3/[3a + (3-a)\lambda]} \cdot \quad (29)$$

The energy-density is given by

$$\rho = \frac{(3-\lambda)}{8\pi m} R_*^{-6}\left[\frac{[3a + (3-a)\lambda]}{3} H_*(1+A)^{(3-\lambda)/3}\,t\right]^{-18/[3a + (3-a)\lambda]} \cdot \quad (30)$$

For energy-density to be positive, we must have $0 < \lambda < 3$. The solution for pressure is obtained by using eqs. (14) and (30) with the limiting value $\gamma = 2a/3$. The Hubble parameter (H) and gravitational constant (G) have the expressions :

$$H = \frac{3}{[3a + (3-a)\lambda]}\,t^{-1}, \quad (31)$$

$$G = B\,t^{2(3-a)(3-\lambda)/[3a + (3-a)\lambda]}, \quad (32)$$

where $$B = \frac{9m}{[3a+(3-a)\lambda]^2} R_*^6\left[\frac{[3a+(3-a)\lambda]}{3} H_*(1+A)^{(3-\lambda)/3}\right]^{18/[3a+(3-a)\lambda]}$$

Using (29) into (21), the scalar potential V is given by

$$V = N\,t^{(3-a)(\lambda-3)/[3a + (3-a)\lambda]}, \quad (33)$$

where $$N = \frac{[3a+(3-a)\lambda]l}{(3-a)(\lambda-3)} R_*^{-3}\left[\frac{[3a+(3-a)\lambda]}{3} H_*(1+A)^{(3-\lambda)/3}\right]^{-9/[3a+(3-a)\lambda]} \cdot$$

Putting the limiting value $\gamma = 2a/3$ for inflationary phase in eq. (28), the asymptotic value of deceleration parameter in the limit $R/R_* \ll 1$, is given by

$$q = \left[3(a-1) + (3-a)\lambda\right] / 3. \tag{34}$$

In order to have expansion, we must have $0 < \lambda < 3$ (since, for inflationary phase, the parameter a must lie in the range $0 \leq a < 1$). We observe that the energy-density is a decreasing function of time. As $t \rightarrow 0$, the energy-density as well as pressure become infinite. Therefore, the model has singularity at $t = 0$. We see that the gravitational 'constant' increases with the age of the Universe which is against to Dirac's hypothesis [2] that the gravitational 'constant' should decrease with time in the expanding universe. The scalar potential decreases as time passes.

Using eqs. (30–33), we find that the eq. (20) is identically satisfied.

3.2. Radiation-dominated phase :

When we consider the radiation-dominated phase ($R \gg R_*$), the first term inside the square bracket on right-hand side of the integral (27) dominates over the second term. Therefore, the solution for scale factor R is given by

$$R = R_* \left[\frac{(6+\lambda)}{3} H_* \left(\frac{1+A}{A}\right)^{(3-\lambda)/3} t\right]^{3/(6+\lambda)}. \tag{35}$$

The energy-density is given by

$$\rho = \frac{(3-\lambda)}{8\pi m} R_*^{-6} \left[\frac{(6+\lambda)}{3} H_* \left(\frac{1+A}{A}\right)^{(3-\lambda)/3} t\right]^{-18/(6+\lambda)}. \tag{36}$$

For energy-density to be positive, we must have $\lambda < 3$. The solution for pressure is obtained by using eqs. (14) and (36) with the limiting value $\gamma = 4/3$. The solution for Hubble parameter and gravitational 'constant' are respectively given by

$$H = \frac{3}{(6+\lambda)} t^{-1} \tag{37}$$

and

$$G = B_1 \, t^{2(3-\lambda)/(6+\lambda)}, \tag{38}$$

where

$$B_1 = \frac{9m}{(6+\lambda)^2} R_*^6 \left[\frac{(6+\lambda)}{3} H_* \left(\frac{1+A}{A}\right)^{(3-\lambda)/3}\right]^{18/(6+\lambda)}.$$

The solution of eq. (21) for this phase is given by

$$V = N_1 \, t^{(\lambda-3)/(6+\lambda)}, \tag{39}$$

where

$$N_1 = \frac{(6+\lambda)l}{(\lambda-3)} R_*^{-3} \left[\frac{(6+\lambda)}{3} H_* \left(\frac{1+A}{A}\right)^{(3-\lambda)/3}\right]^{-9/(6+\lambda)}.$$

Putting the limiting value $\gamma = 4/3$ for radiation-dominated phase in eq. (28), the asymptotic value of deceleration parameter in the limit $R/R_{*} \gg 1$, is given by

$$q = (3+\lambda)/3. \tag{40}$$

In order to have expansion, we must have $0 < \lambda < 3$. The energy-density decreases with time. As $t \to \infty$, the energy-density as well as pressure becomes zero and therefore the model would essentially give an empty universe for large time. The scalar potential decreases with time and tends to zero as $t \to \infty$.

Using eqs. (36–39), we find that the eq. (20) is identically satisfied.

4. Concluding remarks

We have obtained the solutions for spatially homogeneous and isotropic cosmological models with zero-curvature in the presence of perfect fluids and zero-rest-mass scalar fields. A unified description of early evolution of the universe is studied with 'gamma-law' equation of state for two different periods where the gravitational 'constant' is allowed to depend on cosmic time t. The inflationary phase is obtained according to the value of parameter a in eq. (15). The model is an expanding one in each phase for $0 < \lambda < 3$. We also observe that eq. (34) reduces to the solution of a pure radiation phase $R = (2H_{*}t)^{1/2} R_{*}$ for $\lambda = 0$ (see [4]). The solutions obtained in each phase is identically satisfied. The possibility of an increasing G during the transition period is also discussed. A particular case of homogeneous and isotropic solution corresponds to the de Sitter phase when $\lambda = 3$.

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